Problem 1

A line in the plane is called *sunny* if it is **not** parallel to any of the x-axis, the y-axis, and the line $x + y = 0$.

Let $n \geqslant 3$ be a given integer. Determine all nonnegative integers k such that there exist n distinct lines in the plane satisfying both of the following:

- for all positive integers a and b with $a + b \leqslant n + 1$, the point (a, b) is on at least one of the lines; and
- exactly k of the n lines are sunny.

Let $n \geqslant 3$ be a given integer. We want to determine all nonnegative integers k such that there exist n distinct lines in the plane that cover the set of points $P_n = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : a + b \leqslant n + 1\}$, and exactly k of these lines are sunny. A line is sunny if it is not parallel to the x-axis (Horizontal, H), the y-axis (Vertical, V), or the line $x + y = 0$ (Diagonal, D, slope -1). Lines of these three types are called shady.

We will show that the possible values for k are $\{0, 1, 3\}$.

The proof relies on reducing the problem to the specific case where $n = k$ and all lines must be sunny. Let $C(k)$ be the assertion that P_k can be covered by k distinct sunny lines. We define $P_0 = \emptyset$.

1. The Reduction Principle

Let $\mathcal{L}$ be a set of n distinct lines covering P_n. Let k be the number of sunny lines. Let N_V, N_H, N_D be the number of V, H, D lines in $\mathcal{L}$, respectively. Then $N_V + N_H + N_D = n - k$.

Lemma 1 (Structural Lemma). The N_V vertical lines in $\mathcal{L}$ must be $\{x = 1, \dots, x = N_V\}$. The N_H horizontal lines must be $\{y = 1, \dots, y = N_H\}$. The N_D diagonal lines must be $\{x + y = s\}$ for $s = n + 2 - N_D, \dots, n + 1$.

Proof. Consider the column $C_a = P_n \cap \{x = a\}$. We have $|C_a| = n + 1 - a$. Suppose the line $x = a$ is not in $\mathcal{L}$. The points in C_a must be covered by the other lines in $\mathcal{L}$. The N_V vertical lines in $\mathcal{L}$ are distinct from $x = a$, so they do not cover any point in C_a. The remaining $n - N_V$ non-vertical lines each cover at most one point in C_a. Thus, $|C_a| \leq n - N_V$. $n + 1 - a \leq n - N_V$, which implies $a \geq N_V + 1$. By contraposition, if $1 \leq a \leq N_V$, the line $x = a$ must be in $\mathcal{L}$. Since there are exactly N_V vertical lines in $\mathcal{L}$, these must be $\{x = 1, \dots, x = N_V\}$. The argument for horizontal lines is symmetric.

For diagonal lines, consider the anti-diagonal $D_s = P_n \cap \{x + y = s\}$. We have $|D_s| = s - 1$. If $x + y = s$ is not in $\mathcal{L}$, the points in D_s must be covered by the $n - N_D$ lines with slope $\neq -1$. Thus, $s - 1 \leq n - N_D$, so $s \leq n + 1 - N_D$. By contraposition, if $s \geq n + 2 - N_D$, the line $x + y = s$ must be in $\mathcal{L}$.

Theorem 1 (Reduction Theorem). For $n \geq 3$ and $0 \leq k \leq n$, a configuration of n distinct lines covering P_n with exactly k sunny lines exists if and only if $C(k)$ is true.

Proof. ($\Rightarrow$) Let $\mathcal{L}$ be such a configuration. Let N_V, N_H, N_D be the counts of the shady lines, $N_V + N_H + N_D = n - k$. By Lemma 1, the set of shady lines $\mathcal{N}$ is determined. Let R be the set of points in P_n not covered by $\mathcal{N}$. $R = \{(a, b) \in P_n \mid a > N_V, b >$

$N_H, a+b \leq n+1-N_D\}$. The k sunny lines $\mathcal{S} \subset \mathcal{L}$ must cover R. Consider the translation $T(a,b) = (a-N_V, b-N_H) = (a', b')$. If $(a,b) \in R$, then $a' \geq 1, b' \geq 1$. Also, $a'+b' = a+b-(N_V+N_H) \leq (n+1-N_D)-(N_V+N_H) = n+1-(n-k) = k+1$. T maps R bijectively to P_k. The translated lines $T(\mathcal{S})$ cover P_k. Since translation preserves slopes, these k lines are distinct and sunny. Thus $C(k)$ is true.

($\Leftarrow$) Suppose $C(k)$ is true. Let $\mathcal{L}_k$ be k distinct sunny lines covering P_k. Let $N = n-k$. We construct a configuration for P_n. Let $\mathcal{N}$ be the set of N diagonal lines $\{x+y=s \mid s = k+2, \dots, n+1\}$. Let $\mathcal{L} = \mathcal{L}_k \cup \mathcal{N}$. This set has n lines. They are distinct since lines in $\mathcal{L}_k$ have slope $\neq -1$ and lines in $\mathcal{N}$ have slope -1. They cover P_n. If $(a,b) \in P_n$, then $2 \leq a+b \leq n+1$. If $a+b \leq k+1$, then $(a,b) \in P_k$, covered by $\mathcal{L}_k$. If $k+2 \leq a+b \leq n+1$, then (a,b) is covered by $\mathcal{N}$. The configuration has exactly k sunny lines.

2. Analysis of the Core Problem $C(k)$

We determine the values of $k \geq 0$ for which P_k can be covered by k distinct sunny lines.

1. $k=0$. $P_0 = \emptyset$. Covered by 0 lines. $C(0)$ is true. 2. $k=1$. $P_1 = \{(1,1)\}$. Covered by $y=x$ (slope 1, sunny). $C(1)$ is true. 3. $k=2$. $P_2 = \{(1,1),(1,2),(2,1)\}$. The lines connecting any pair of these points are $x=1$ (V), $y=1$ (H), or $x+y=3$ (D). All are shady. A sunny line can cover at most one point of P_2. To cover the 3 points, we need at least 3 sunny lines. Thus $C(2)$ is false.

4. $k \geq 3$. Let T_k be the convex hull of P_k. T_k is the triangle with vertices $V_1 = (1,1), V_2 = (1,k), V_3 = (k,1)$. The edges of T_k lie on the lines $x=1$ (V), $y=1$ (H), and $x+y=k+1$ (D). These lines are shady.

Let B_k be the set of points in P_k lying on the boundary of T_k. Each edge contains k points. Since the vertices are distinct (as $k \geq 2$), the total number of points on the boundary is $|B_k| = 3k-3$.

Suppose P_k is covered by k sunny lines $\mathcal{L}_k$. These lines must cover B_k. Let $L \in \mathcal{L}_k$. Since L is sunny, it does not coincide with the lines containing the edges of T_k. A line that does not contain an edge of a convex polygon intersects the boundary of the polygon at most at two points. Thus, $|L \cap B_k| \leq 2$. The total coverage of B_k by $\mathcal{L}_k$ is at most $2k$. We must have $|B_k| \leq 2k$. $3k-3 \leq 2k$, which implies $k \leq 3$.

Since we assumed $k \geq 3$, we must have $k=3$.

5. $k=3$. We verify $C(3)$. P_3 consists of 6 points: $(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)$. We provide a covering with 3 sunny lines: $L_1 : y=x$ (slope 1). Covers $(1,1),(2,2)$. $L_2 : 2x+y=5$ (slope -2). Covers $(1,3),(2,1)$. $L_3 : x+2y=5$ (slope $-1/2$). Covers $(1,2),(3,1)$. These lines are sunny and cover P_3. $C(3)$ is true.

3. Conclusion

The property $C(k)$ is true if and only if $k \in \{0,1,3\}$. By the Reduction Theorem, for a given $n \geq 3$, a configuration with k sunny lines exists if and only if $C(k)$ is true and $k \leq n$. Since $n \geq 3$, the condition $k \leq n$ is satisfied for all $k \in \{0,1,3\}$.

The possible values for k are 0, 1, and 3.

Problem 2

Let Ω and Γ be circles with centres M and N, respectively, such that the radius of Ω is less than the radius of Γ. Suppose circles Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD. Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN.

Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF.

(The *orthocentre* of a triangle is the point of intersection of its altitudes.)

Complete Proof

1. Identification of P as the Excenter of $\triangle AMN$.

Let R_1 and R_2 be the radii of Ω (center M) and Γ (center N) respectively, with $R_1 < R_2$. P is the circumcenter of $\triangle ACD$, so $PA = PC$. Since $A, C \in \Omega$, $MA = MC = R_1$. Thus PM is the perpendicular bisector of AC and bisects $\angle AMC$. The points C, M, N, D are collinear in this order. This implies that the ray MC is opposite to the ray MN. Therefore, $\angle AMC$ and $\angle AMN$ are supplementary. $\angle AMC$ is the exterior angle of $\triangle AMN$ at M. Since PM bisects $\angle AMC$, PM is the external angle bisector of $\triangle AMN$ at M.

Similarly, $PA = PD$ and $NA = ND = R_2$. PN is the perpendicular bisector of AD and bisects $\angle AND$. Since M, N, D are in order, the ray ND is opposite to the ray NM. Thus, $\angle AND$ is the exterior angle of $\triangle AMN$ at N. PN is the external angle bisector of $\triangle AMN$ at N.

Therefore, P is the excenter of $\triangle AMN$ opposite to A. Consequently, the line AP is the internal angle bisector of $\angle MAN$. Let $\angle MAN = 2\phi$. Since the circles intersect at two distinct points A and B, $\triangle AMN$ is non-degenerate, so $0 < 2\phi < 180°$, i.e., $0 < \phi < 90°$.

2. Determining $\angle EBF$.

By symmetry with respect to the line MN, $\triangle BMN \cong \triangle AMN$. Thus $\angle MBN = \angle MAN = 2\phi$.

We use directed angles modulo 180°. Let $T_M(B)$ and $T_N(B)$ be the tangents to Ω and Γ at B, respectively. Since $T_M(B) \perp MB$ and $T_N(B) \perp NB$, we have $\measuredangle(T_M(B), T_N(B)) = \measuredangle(MB, NB)$.

By the Tangent-Chord Theorem: In Ω, $\measuredangle(T_M(B), BE) = \measuredangle(AB, AE)$. In Γ, $\measuredangle(T_N(B), BF) = \measuredangle(AB, AF)$. Since A, E, F are collinear on the line AP, the lines AE and AF are the same. Thus $\measuredangle(AB, AE) = \measuredangle(AB, AF)$.

We compute $\measuredangle(BE, BF)$: $\measuredangle(BE, BF) = \measuredangle(BE, T_M(B)) + \measuredangle(T_M(B), T_N(B)) + \measuredangle(T_N(B), BF)$ $= -\measuredangle(AB, AE) + \measuredangle(MB, NB) + \measuredangle(AB, AF) = \measuredangle(MB, NB)$. Thus, the geometric angle $\angle EBF = \angle MBN = 2\phi$.

Since $R_1 \neq R_2$, $\triangle AMN$ is not isosceles, so AP (the angle bisector) is distinct from the altitude from A. Since AB is perpendicular to MN, AB is the altitude line. Thus B is not on AP. Also $R_1 \neq R_2$ implies $E \neq F$. Thus $\triangle BEF$ is non-degenerate. Let Σ be its circumcircle.

3. Introduction of the Auxiliary Point V and its properties.

Let V be the point such that $AMVN$ is a parallelogram. We use vectors originating from A. $\vec{AV} = \vec{AM} + \vec{AN}$.

We calculate the lengths of AE and AF. In $\triangle AME$, $MA = ME = R_1$ and $\angle MAE = \phi$. Thus $AE = 2R_1 \cos\phi$. Similarly, $AF = 2R_2 \cos\phi$. Since $R_1 < R_2$ and $\cos\phi > 0$, $AE < AF$. A, E, F are collinear in this order on AP. $EF = AF - AE = 2(R_2 - R_1)\cos\phi$.

We calculate the distances VE and VF. $\vec{VE} = \vec{AE} - \vec{AV} = \vec{AE} - (\vec{AM} + \vec{AN})$. $VE^2 = AE^2 + AM^2 + AN^2 - 2\vec{AE}\cdot\vec{AM} - 2\vec{AE}\cdot\vec{AN} + 2\vec{AM}\cdot\vec{AN}$. $AM = R_1, AN = R_2$. $\angle MAN = 2\phi$. $\angle MAE = \angle NAE = \phi$. $\vec{AE}\cdot\vec{AM} = AE\cdot R_1\cos\phi = 2R_1^2\cos^2\phi$. $\vec{AE}\cdot\vec{AN} = AE\cdot R_2\cos\phi = 2R_1R_2\cos^2\phi$. $\vec{AM}\cdot\vec{AN} = R_1R_2\cos(2\phi) = R_1R_2(2\cos^2\phi - 1)$.

$VE^2 = (2R_1\cos\phi)^2 + R_1^2 + R_2^2 - 4R_1^2\cos^2\phi - 4R_1R_2\cos^2\phi + 2R_1R_2(2\cos^2\phi - 1)$. $VE^2 = R_1^2 + R_2^2 - 4R_1R_2\cos^2\phi + 4R_1R_2\cos^2\phi - 2R_1R_2$. $VE^2 = R_1^2 + R_2^2 - 2R_1R_2 = (R_2 - R_1)^2$. So $VE = R_2 - R_1$. A similar calculation shows $VF = R_2 - R_1$. Thus $VE = VF$.

4. V lies on the circumcircle Σ.

We calculate $\angle EVF$ using the Law of Cosines in the isosceles triangle $\triangle EVF$. $EF^2 = VE^2 + VF^2 - 2VE\cdot VF\cos(\angle EVF) = 2VE^2(1 - \cos(\angle EVF))$. $(2(R_2 - R_1)\cos\phi)^2 = 2(R_2 - R_1)^2(1 - \cos(\angle EVF))$. $4\cos^2\phi = 2(1 - \cos(\angle EVF))$. $\cos(\angle EVF) = 1 - 2\cos^2\phi = -\cos(2\phi)$. Since $2\phi \in (0, 180°)$, $\angle EVF = 180° - 2\phi$.

We have $\angle EBF + \angle EVF = 2\phi + (180° - 2\phi) = 180°$. To conclude that $BEVF$ is cyclic, we must verify that B and V lie on opposite sides of the line AP. We set up a coordinate system with A at the origin $(0, 0)$ and AP along the positive x-axis. We can orient it such that $M = (R_1\cos\phi, R_1\sin\phi)$ and $N = (R_2\cos\phi, -R_2\sin\phi)$. Then $V = M + N$ has y-coordinate $y_V = (R_1 - R_2)\sin\phi$. Since $R_1 < R_2$ and $\phi > 0$, $y_V < 0$.

B is the reflection of A across the line MN. The line MN has the equation $y - y_M = m(x - x_M)$, where the slope is $m = \frac{-(R_1+R_2)\sin\phi}{(R_2-R_1)\cos\phi}$. The y-intercept b (intersection with the axis perpendicular to AP through A) is $y_M - mx_M$. $b = R_1\sin\phi - mR_1\cos\phi = R_1\sin\phi + \frac{R_1(R_1+R_2)\sin\phi}{R_2-R_1} = \frac{2R_1R_2\sin\phi}{R_2-R_1}$. Since $R_i > 0$ and $\sin\phi > 0$, $b > 0$. The line MN passes "above" A with respect to the y-axis. The reflection B of $A(0, 0)$ across the line $y = mx + b$ has y-coordinate $y_B = 2b/(m^2 + 1) > 0$. Since $y_V < 0$ and $y_B > 0$, V and B are on opposite sides of AP. Thus, $BEVF$ is cyclic, and V lies on Σ.

5. The Orthocenter H and the Tangency Condition.

Let I be the incenter of $\triangle AMN$. Since P is the excenter opposite to A, the points A, I, P are collinear on the line AP. The internal bisector MI and the external bisector MP at M are perpendicular. Similarly, $NI \perp NP$. Thus, the quadrilateral $IMPN$ is cyclic. This circle is the circumcircle of $\triangle PMN$. Let O be its center. IP is the diameter, so O is the midpoint of IP.

H is the orthocenter of $\triangle PMN$. By Sylvester's theorem relating the circumcenter O and the orthocenter H, we have (using vectors originating from A): $\vec{AH} = \vec{AP} + \vec{AM} + \vec{AN} - 2\vec{AO}$. By definition of V, $\vec{AV} = \vec{AM} + \vec{AN}$. $\vec{AH} = \vec{AP} + \vec{AV} - 2\vec{AO}$. The vector from V

to H is $\vec{VH} = \vec{AH} - \vec{AV} = \vec{AP} - 2\vec{AO}$. Since O is the midpoint of IP, $2\vec{AO} = \vec{AI} + \vec{AP}$. $\vec{VH} = \vec{AP} - (\vec{AI} + \vec{AP}) = -\vec{AI} = \vec{IA}$.

Since I and A lie on the line AP, the vector $\vec{IA}$ is parallel to AP. Thus, the line segment VH is parallel to AP. The line through H parallel to AP is the line VH.

We must show that the line VH is tangent to Σ. Since $V \in \Sigma$ (Step 4), it suffices to show that VH is perpendicular to the radius at V. Let O_Σ be the center of Σ. We need to show $VH \perp O_\Sigma V$. Since $VH \parallel AP$, we need $AP \perp O_\Sigma V$. The points E, F lie on AP. In Step 3, we proved $VE = VF$. Thus V lies on the perpendicular bisector of the chord EF. O_Σ also lies on this bisector. Therefore, the line $O_\Sigma V$ is the perpendicular bisector of EF. Thus $O_\Sigma V \perp EF$. Since EF lies on AP, $O_\Sigma V \perp AP$.

We conclude that $VH \perp O_\Sigma V$. Therefore, the line VH, which is the line through H parallel to AP, is tangent to the circumcircle of triangle BEF at V.

Problem 3

Let $\mathbb{N}$ denote the set of positive integers. A function $f\colon \mathbb{N} \to \mathbb{N}$ is said to be *bonza* if

$$f(a) \text{ divides } b^a - f(b)^{f(a)}$$

for all positive integers a and b.

Determine the smallest real constant c such that $f(n) \leqslant cn$ for all bonza functions f and all positive integers n.

We want to determine the smallest real constant c such that $f(n) \leqslant cn$ for all bonza functions $f\colon \mathbb{N} \to \mathbb{N}$ and all $n \in \mathbb{N}$. A function f is bonza if $f(a) \mid b^a - f(b)^{f(a)}$ for all $a, b \in \mathbb{N}$. Let $P(a, b)$ denote this assertion.

We will show that the smallest constant is $c = 4$.

Part 1: Properties and Classification of Bonza Functions

First, we establish some basic properties. $P(a, a)$ implies $f(a) \mid a^a$. $P(1, 1)$ implies $f(1) = 1$. Let S be the set of primes p such that $f(p) > 1$. Since $f(p) \mid p^p$, if $p \in S$, then $f(p) = p^k$ for some $k \geq 1$.

Lemma 1: If $p \in S$, then $f(b) \equiv b \pmod{p}$ for all $b \in \mathbb{N}$. *Proof:* $P(p, b) \implies f(p) \mid b^p - f(b)^{f(p)}$. Since $p \in S$, $p \mid f(p)$. Thus $b^p \equiv f(b)^{f(p)} \pmod{p}$. By Fermat's Little Theorem (FLT), $b^p \equiv b \pmod{p}$. Since $f(p)$ is a power of p, applying FLT repeatedly yields $f(b)^{f(p)} \equiv f(b) \pmod{p}$. Thus, $b \equiv f(b) \pmod{p}$.

Lemma 2 (Classification): The set S is either the set of all primes $\mathbb{P}$, the empty set $\emptyset$, or the singleton set $\{2\}$. *Proof:* Case 1: S is infinite. By Lemma 1, for any $b \in \mathbb{N}$, $f(b) - b$ is divisible by every prime in S. Since S is infinite, $f(b) - b = 0$, so $f(b) = b$ for all b. Then $f(p) = p > 1$ for all primes p, so $S = \mathbb{P}$.

Case 2: S is finite. Let $M = \prod_{p \in S} p$. (If $S = \emptyset$, $M = 1$). Let q be a prime not in S. Then $f(q) = 1$. If S is non-empty, for any $p \in S$, Lemma 1 gives $1 = f(q) \equiv q \pmod{p}$. Thus $q \equiv 1 \pmod{M}$.

Suppose S is finite and non-empty. Then $M \geq 2$. Suppose $M > 2$. Consider $A = M - 1$. Since $M > 2$, $1 < A < M$. We have $\gcd(A, M) = 1$. Let q_0 be any prime factor of A. Then $q_0 \nmid M$, so $q_0 \notin S$. Thus $q_0 \equiv 1 \pmod{M}$. This implies $M \mid q_0 - 1$, so $M \leq q_0 - 1$. Since $q_0 \mid A$, $q_0 \leq A = M - 1$. Combining these gives $M \leq q_0 - 1 \leq (M-1) - 1 = M - 2$. $M \leq M - 2$, which is a contradiction. Therefore, if S is finite and non-empty, we must have $M = 2$. This means $S = \{2\}$. If S is empty, $M = 1$.

Part 2: Establishing the Upper Bound $c \leq 4$

We analyze the three cases from Lemma 2.

Case 1: $S = \mathbb{P}$. We found $f(n) = n$. Then $f(n)/n = 1$.

Case 2: $S = \emptyset$. $f(p) = 1$ for all primes p. Let $n \in \mathbb{N}$. If $f(n) > 1$, let q be a prime factor of $f(n)$. Since $f(n) \mid n^n$, $q \mid n$. $P(n, q) \implies f(n) \mid q^n - f(q)^{f(n)}$. Since $q \notin S$, $f(q) = 1$. So $f(n) \mid q^n - 1$. Since $q \mid f(n)$, $q \mid q^n - 1$. As $q \mid n$, $q \mid q^n$. Thus $q \mid 1$. Contradiction. So $f(n) = 1$ for all n. $f(n)/n \leq 1$.

Case 3: $S = \{2\}$. $f(2) > 1$, and $f(p) = 1$ for all odd primes p. First, we show $f(n)$ is a power of 2 for all n. Let q be an odd prime factor of $f(n)$. Then $q \mid n$. $f(q) = 1$. $P(n, q) \implies f(n) \mid q^n - f(q)^{f(n)} = q^n - 1$. Since $q \mid f(n)$, $q \mid q^n - 1$. This is impossible as $q \mid n$ implies $q \mid q^n$. Thus $f(n)$ is a power of 2.

If n is odd, $f(n) \mid n^n$ (odd). So $f(n) = 1$.

If n is even. Let $n = 2^k m$, where $k = v_2(n) \geq 1$ and m is odd. Let $f(n) = 2^e$. Let b be any odd integer. $f(b) = 1$. $P(n, b) \implies f(n) \mid b^n - f(b)^{f(n)} = b^n - 1$. So $2^e \mid b^n - 1$. Thus $e \leq \min_{b \text{ odd}} v_2(b^n - 1)$.

We need the following lemma to analyze the 2-adic valuation.

Lemma 3: Let X be an odd integer and $K \geq 1$ an integer. Then $v_2(X^{2^K} - 1) = v_2(X^2 - 1) + K - 1$. *Proof:* We use induction on K. Base case $K = 1$: $v_2(X^2 - 1) = v_2(X^2 - 1) + 1 - 1$. Inductive step: Assume it holds for $K \geq 1$. We check $K + 1$. $v_2(X^{2^{K+1}} - 1) = v_2((X^{2^K} - 1)(X^{2^K} + 1))$. Since X is odd, $X^2 \equiv 1 \pmod 8$. Since $K \geq 1$, $X^{2^K} = (X^2)^{2^{K-1}} \equiv 1^{2^{K-1}} = 1 \pmod 8$. Thus $X^{2^K} + 1 \equiv 2 \pmod 8$, so $v_2(X^{2^K} + 1) = 1$. $v_2(X^{2^{K+1}} - 1) = v_2(X^{2^K} - 1) + 1 = (v_2(X^2 - 1) + K - 1) + 1 = v_2(X^2 - 1) + K$.

Now we analyze $v_2(b^n - 1) = v_2(b^{2^k m} - 1)$. Let $X = b^m$. Since b, m are odd, X is odd. By Lemma 3 (with $K = k$), $v_2(b^n - 1) = v_2(X^{2^k} - 1) = v_2(X^2 - 1) + k - 1$. We want to minimize this over odd b. $X^2 - 1 = b^{2m} - 1$. Since b^m is odd, $(b^m)^2 \equiv 1 \pmod 8$, so $v_2(b^{2m} - 1) \geq 3$. The minimum is achieved when $b = 3$. We calculate $v_2(3^{2m} - 1) = v_2(9^m - 1)$. $9^m - 1 = (9 - 1)(9^{m-1} + \cdots + 1)$. The second factor is a sum of m odd terms. Since m is odd, the sum is odd. $v_2(9^m - 1) = v_2(8) = 3$. Thus, $\min_{b \text{ odd}} v_2(b^n - 1) = 3 + (k - 1) = k + 2$. So $e \leq k + 2$.

The ratio is $\frac{f(n)}{n} = \frac{2^e}{2^k m} \leq \frac{2^{k+2}}{2^k m} = \frac{4}{m}$. Since $m \geq 1$, $f(n)/n \leq 4$.

In all cases, $f(n) \leq 4n$ for all bonza functions f. Thus $c \leq 4$.

Part 3: Construction and Lower Bound $c \geq 4$

We construct a bonza function $g(n)$ that achieves the bound 4. Define $g(n)$ as follows:

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 16 & \text{if } n = 4 \\ 2 & \text{if } n \text{ is even and } n \neq 4 \end{cases}$$

For $n = 4$, $g(4) = 16$, so $g(4)/4 = 4$. If g is bonza, then $c \geq 4$.

We verify that g is bonza. We check $g(a) \mid b^a - g(b)^{g(a)}$.

Case 1: a is odd. $g(a) = 1$. The condition holds trivially.

Case 2: $a = 4$. $g(4) = 16$. We need $16 \mid b^4 - g(b)^{16}$. If b is odd, $g(b) = 1$. We need $16 \mid b^4 - 1$. For any odd b, $b^2 \pmod{16}$ is in $\{1, 9\}$. So $b^4 \equiv 1 \pmod{16}$. If b is even, $v_2(b) \geq 1$. $v_2(b^4) \geq 4$. So $16 \mid b^4$. $g(b)$ is either 2 or 16. $g(b)$ is even. $v_2(g(b)^{16}) \geq 16$. So $16 \mid g(b)^{16}$. The condition holds.

Case 3: a is even and $a \neq 4$. $g(a) = 2$. $a \geq 2$. We need $2 \mid b^a - g(b)^2$. If b is odd, b^a is odd. $g(b) = 1$. $b^a - g(b)^2$ is odd-1=even. If b is even. b^a is even (since $a \geq 2$). $g(b) \in \{2, 16\}$. $g(b)$ is even, so $g(b)^2$ is even. $b^a - g(b)^2$ is even. The condition holds.

The function $g(n)$ is bonza.

Conclusion

We have proved that $f(n) \leq 4n$ for all bonza functions f, so $c \leq 4$. We have constructed a bonza function $g(n)$ such that $g(4)/4 = 4$, so $c \geq 4$. Therefore, the smallest real constant c is 4.

Problem 4

A *proper divisor* of a positive integer N is a positive divisor of N other than N itself.

The infinite sequence $a_1, a_2, \ldots$ consists of positive integers, each of which has at least three proper divisors. For each $n \geqslant 1$, the integer a_{n+1} is the sum of the three largest proper divisors of a_n.

Determine all possible values of a_1.

We want to determine all possible values of a_1 such that the sequence defined by $a_{n+1} = S(a_n)$ is an infinite sequence of positive integers, where $S(N)$ is the sum of the three largest proper divisors of N, and each a_n has at least three proper divisors (i.e., the total number of divisors $\tau(a_n) \geq 4$).

Let $R(N) = S(N)/N$. If $\tau(N) \geq 5$, and $1 = d_1 < d_2 < d_3 < d_4$ are the smallest divisors of N, then $R(N) = \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4}$.

Step 1: Proving that a_n is even for all n.

Suppose N is odd and $\tau(N) \geq 4$. The divisors of N are odd. If $\tau(N) \geq 5$, $d_2 \geq 3, d_3 \geq 5, d_4 \geq 7$. $R(N) \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{71}{105} < 1$. If $\tau(N) = 4$, $N = p^3$ or $N = pq$ for odd primes $p < q$. The proper divisors are $1, p, p^2$ or $1, p, q$. $S(p^3) = 1 + p + p^2$. For $p \geq 3$, $p^3 - (1 + p + p^2) = p^2(p-1) - p - 1 \geq 9(2) - 3 - 1 = 14 > 0$. $S(pq) = 1 + p + q$. For $p \geq 3, q \geq 5$, $pq - (1 + p + q) = (p-1)(q-1) - 2 \geq 2 \cdot 4 - 2 = 6 > 0$. In all cases, $S(N) < N$. Furthermore, the three largest proper divisors are odd, so their sum $S(N)$ is odd.

If a_n were odd for some n. Since $\tau(a_n) \geq 4$ by the problem statement, $a_{n+1} = S(a_n) < a_n$ and a_{n+1} is odd. By induction, $(a_k)_{k \geq n}$ would be a strictly decreasing infinite sequence of positive integers. This contradicts the Well-Ordering Principle. Thus, a_n is even for all n.

Step 2: Proving that a_n is divisible by 3 for all n.

Suppose N is even, $\tau(N) \geq 4$, and $3 \nmid N$. $d_2 = 2$. Since $3 \nmid N$, $d_3 \geq 4$. If $\tau(N) \geq 5$, $d_4 \geq 5$. $R(N) \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < 1$. If $\tau(N) = 4$. $N = 8$ or $N = 2p$ (prime $p \geq 5$). $S(8) = 7 < 8$. $S(2p) = p + 3 < 2p$. In all cases, $S(N) < N$.

We prove a lemma: Lemma: Let N be even, $\tau(N) \geq 4$, and $3 \nmid N$. If $3 | S(N)$, then $S(N)$ is odd. Proof: If $\tau(N) = 4$, $S(8) = 7$, $S(2p) = p + 3$. Since $3 \nmid p$, $3 \nmid p + 3$. So $3 \nmid S(N)$. The implication holds vacuously. If $\tau(N) \geq 5$. $R(N) = \frac{1}{2} + \frac{1}{d_3} + \frac{1}{d_4}$. Since $3 \nmid N$, $3 \nmid d_i$. If $3 | S(N)$, since $3 \nmid N$, we must have $v_3(R(N)) > 0$. $R(N) = \frac{d_3 d_4 + 2d_3 + 2d_4}{2d_3 d_4}$. The denominator is not divisible by 3. The numerator $X = d_3 d_4 + 2d_3 + 2d_4$ must be divisible by 3. $X \equiv d_3 d_4 - d_3 - d_4 \equiv (d_3 - 1)(d_4 - 1) - 1 \pmod 3$. $X \equiv 0 \implies (d_3 - 1)(d_4 - 1) \equiv 1 \pmod 3$. This requires $d_3 \equiv 2$ and $d_4 \equiv 2 \pmod 3$. If $4 | N$. Since $3 \nmid N$, the divisors start $1, 2, 4$. So $d_3 = 4$. But $4 \equiv 1 \pmod 3$, contradicting $d_3 \equiv 2 \pmod 3$. Thus $v_2(N) = 1$. $N = 2M$ with M odd, $3 \nmid M$. Let p be the smallest prime factor of M ($p \geq 5$). $d_3 = p$. We need $p \equiv 2 \pmod 3$. d_4 is the next smallest divisor. $2p \equiv 2(2) = 4 \equiv 1 \pmod 3$. Since $d_4 \equiv 2 \pmod 3$, $d_4 \neq 2p$. So d_4 must be the next smallest divisor of M, call it m_3. d_4 is odd. $S(N) = N/2 + N/p + N/d_4 = M + 2M/p + 2M/d_4$. Since M is odd and p, d_4 are odd divisors of M, M/p and M/d_4 are odd integers. $S(N) = \text{Odd} + \text{Even} + \text{Even} = \text{Odd}$. The lemma is proved.

Now, suppose $3 \nmid a_n$ for some n. We know a_n is even and $a_{n+1} = S(a_n) < a_n$. If $3|a_{n+1}$, by the Lemma applied to a_n, a_{n+1} must be odd. This contradicts Step 1. Thus $3 \nmid a_{n+1}$. By induction, $(a_k)_{k\geq n}$ is a strictly decreasing infinite sequence of positive integers. Contradiction. Therefore, $3|a_n$ for all n.

Combining Step 1 and Step 2, $6|a_n$ for all n. Note that $6|N$ implies $\tau(N) \geq 4$.

Step 3: Analyzing the dynamics when $6|N$.

If $6|N$, the smallest divisors are $1, 2, 3$. The fourth smallest divisor d_4 must be 4, 5, or 6. $R(N) = \frac{1}{2} + \frac{1}{3} + \frac{1}{d_4} = \frac{5}{6} + \frac{1}{d_4}$. (This holds even if $\tau(N) = 4$, i.e., $N = 6$, where $S(6) = 6, R(6) = 1$, and d_4 is formally $N = 6$).

We identify three regimes: Regime A (Growth): $d_4 = 4$. Occurs if $12|N$. $R(N) = 13/12$. Regime B (Boost): $d_4 = 5$. Occurs if $30|N$ and $4 \nmid N$ ($v_2(N) = 1$). $R(N) = 31/30$. Regime C (Fixed Point): $d_4 = 6$. Occurs if $6|N, 4 \nmid N, 5 \nmid N$. $R(N) = 1$.

Step 4: Evolution of the sequence and constraints on a_1.

If $a_n \in B$. $v_2(a_n) = 1$. $a_{n+1} = (31/30)a_n$. $v_2(a_{n+1}) = v_2(a_n) + v_2(31/30) = 1 - 1 = 0$. a_{n+1} is odd. This contradicts Step 1. Thus, the sequence must remain in A $\cup$ C.

If $a_n \in A$. $a_{n+1} = (13/12)a_n$. $v_2(a_{n+1}) = v_2(a_n) - 2$. $v_3(a_{n+1}) = v_3(a_n) - 1$. Since $6|a_k$ for all k, $v_2(a_k) \geq 1$ and $v_3(a_k) \geq 1$. As the valuations decrease in Regime A, the sequence cannot stay in A indefinitely. It must eventually reach Regime C and stabilize there ($a_{n+1} = a_n$).

In Regimes A ($R = 13/12$) and C ($R = 1$), $v_5(R(N)) = 0$. Thus $v_5(a_n)$ is constant. Let L be the stable value in C. By definition of C, $5 \nmid L$. So $v_5(L) = 0$. Therefore, $v_5(a_1) = 0$.

Step 5: Characterization of a_1.

Let $K \geq 0$ be the number of steps the sequence spends in Regime A before reaching Regime C. $a_1, \ldots, a_K \in A$ (if $K \geq 1$) and $a_{K+1} \in C$. Since $5 \nmid a_1$, $5 \nmid a_n$ for all n.

Let $A = v_2(a_1)$ and $B = v_3(a_1)$. $a_{K+1} = (13/12)^K a_1$. $v_2(a_{K+1}) = A - 2K$. $v_3(a_{K+1}) = B - K$. Since $a_{K+1} \in C$, we must have $v_2(a_{K+1}) = 1$ (as $6|a_{K+1}$ and $4 \nmid a_{K+1}$) and $v_3(a_{K+1}) \geq 1$. $A - 2K = 1 \implies A = 2K + 1$. $B - K \geq 1 \implies B \geq K + 1$.

We verify that these conditions are sufficient. We must ensure $a_i \in A$ for $1 \leq i \leq K$. This means $12|a_i$. For $1 \leq i \leq K$: $v_2(a_i) = A - 2(i-1) = 2K + 1 - 2i + 2 = 2(K - i) + 3$. Since $i \leq K$, $v_2(a_i) \geq 3$. $v_3(a_i) = B - (i-1) \geq (K+1) - (i-1) = K - i + 2$. Since $i \leq K$, $v_3(a_i) \geq 2$. Thus $2^3 \cdot 3^2 = 72$ divides a_i. This implies $12|a_i$, so $a_i \in A$. This also ensures that $a_{i+1} = (13/12)a_i$ is an integer. The sequence is valid.

We express the possible values of a_1. $a_1 = 2^{2K+1}3^B M$, where $K \geq 0$, $B \geq K+1$, and M is a positive integer such that $\gcd(M, 30) = 1$ (since $v_5(a_1) = 0$). We rewrite this as: $a_1 = (2^{2K+1}3^{K+1}) \cdot (3^{B-(K+1)}M)$. $2^{2K+1}3^{K+1} = (2 \cdot 4^K) \cdot (3 \cdot 3^K) = 6 \cdot (12^K)$. Let $J = 3^{B-K-1}M$. J is a positive integer. Since $\gcd(M, 30) = 1$, J is not divisible by 2 or 5. That is, $\gcd(J, 10) = 1$. Conversely, any positive integer J such that $\gcd(J, 10) = 1$ can be represented in this form for a given K (by taking $B = K+1+v_3(J)$ and $M = J/3^{v_3(J)}$).

The set of all possible values of a_1 consists of the integers of the form $6J \cdot 12^K$, where $K \geq 0$ is an integer and J is a positive integer such that $\gcd(J, 10) = 1$.

Problem 5

Alice and Bazza are playing the *inekoalaty game*, a two-player game whose rules depend on a positive real number λ which is known to both players. On the n^{th} turn of the game (starting with $n = 1$) the following happens:

- If n is odd, Alice chooses a nonnegative real number x_n such that

$$x_1 + x_2 + \cdots + x_n \leqslant \lambda n.$$

- If n is even, Bazza chooses a nonnegative real number x_n such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant n.$$

If a player cannot choose a suitable number x_n, the game ends and the other player wins. If the game goes on forever, neither player wins. All chosen numbers are known to both players.

Determine all values of λ for which Alice has a winning strategy and all those for which Bazza has a winning strategy.

We determine the values of λ for which Alice has a winning strategy and those for which Bazza has a winning strategy. Let $S_n = \sum_{i=1}^n x_i$ and $Q_n = \sum_{i=1}^n x_i^2$. Alice (A) plays at odd n, ensuring $x_n \geq 0$ and $S_n \leq \lambda n$. Bazza (B) plays at even n, ensuring $x_n \geq 0$ and $Q_n \leq n$. The critical value for λ is $\frac{1}{\sqrt{2}}$.

Case 1: $0 < \lambda < \frac{1}{\sqrt{2}}$. Bazza has a winning strategy.

Let $\delta = \sqrt{2} - 2\lambda$. Since $\lambda < \frac{1}{\sqrt{2}}$, we have $\delta > 0$.

Bazza's strategy (B-MaxQ) is to ensure $Q_{2k} = 2k$ at every turn $n = 2k$. This requires choosing $x_{2k} = \sqrt{2k - Q_{2k-1}}$. This is feasible if $Q_{2k-1} \leq 2k$.

Let C_k be the budget available to Alice at the start of turn $2k-1$: $C_k = \lambda(2k-1) - S_{2k-2}$ (with $S_0 = 0$). Alice must choose $x_{2k-1} \in [0, C_k]$. If $C_k < 0$, Alice loses immediately.

We analyze the evolution of C_k, assuming the game continues and Bazza follows B-MaxQ. $C_{k+1} = \lambda(2k+1) - S_{2k} = C_k + 2\lambda - (x_{2k-1} + x_{2k})$.

If Bazza successfully follows B-MaxQ up to turn $2k$, then $Q_{2k} = 2k$ and $Q_{2k-2} = 2k-2$. Thus, $x_{2k-1}^2 + x_{2k}^2 = Q_{2k} - Q_{2k-2} = 2$. Since $x_i \geq 0$, $(x_{2k-1} + x_{2k})^2 = 2 + 2x_{2k-1}x_{2k} \geq 2$, so $x_{2k-1} + x_{2k} \geq \sqrt{2}$.

Therefore, $C_{k+1} \leq C_k + 2\lambda - \sqrt{2} = C_k - \delta$.

We must verify that B-MaxQ is always feasible as long as the game continues (i.e., $C_k \geq 0$). We proceed by induction. $C_1 = \lambda$. Since $\delta > 0$, if $C_k \geq 0$, the sequence C_k is strictly decreasing. Thus $C_k \leq C_1 = \lambda$. Since $\lambda < 1/\sqrt{2}$. Alice must choose $x_{2k-1} \leq C_k < 1/\sqrt{2}$. If Bazza maintained $Q_{2k-2} = 2k-2$, then $Q_{2k-1} = Q_{2k-2} + x_{2k-1}^2 = 2k - 2 + x_{2k-1}^2 < 2k - 2 + 1/2 = 2k - 3/2$. Since $Q_{2k-1} < 2k$, Bazza can choose x_{2k} to achieve $Q_{2k} = 2k$. B-MaxQ is always feasible.

Since $C_{k+1} \le C_k - \delta$, the budget decreases by at least δ in each round pair. $C_k \le C_1 - (k-1)\delta = \lambda - (k-1)\delta$. Since λ is fixed and $\delta > 0$, there exists an integer K such that $(K-1)\delta > \lambda$. For this K, $C_K < 0$. At turn $2K-1$, Alice needs to choose $x_{2K-1} \ge 0$ such that $x_{2K-1} \le C_K$. Since $C_K < 0$, no such choice exists. Bazza wins.

Case 2: $\lambda > \frac{1}{\sqrt{2}}$. Alice has a winning strategy.

Consider the function $h(K) = \frac{K\sqrt{2}}{2K-1}$ for $K \ge 1$. $h(K)$ is strictly decreasing and $\lim_{K\to\infty} h(K) = 1/\sqrt{2}$. Since $\lambda > 1/\sqrt{2}$, there exists an integer $K \ge 1$ such that $\lambda > h(K)$. This implies $L = \lambda(2K-1) > K\sqrt{2}$.

Alice's strategy (A-Spike-K): Play $x_{2i-1} = 0$ for $i = 1, \ldots, K-1$. At turn $2K-1$, play the maximum possible value.

First, we verify the feasibility. For $i < K$, Alice plays $x_{2i-1} = 0$. She needs $S_{2i-1} = S_{2i-2} \le \lambda(2i-1)$. Bazza is constrained by $Q_{2i-2} \le 2(i-1)$. By the QM-AM inequality (or Cauchy-Schwarz), $S_{2i-2} \le \sqrt{(i-1)Q_{2i-2}} \le \sqrt{(i-1)2(i-1)} = (i-1)\sqrt{2}$. We check the constraint: $(i-1)\sqrt{2} \le \lambda(2i-1)$, or $\lambda \ge \frac{(i-1)\sqrt{2}}{2i-1}$. The RHS is an increasing sequence converging to $1/\sqrt{2}$. Since $\lambda > 1/\sqrt{2}$, the strategy is feasible.

Now we analyze the outcome. Let $N = K-1$. Bazza has made N moves $y_i = x_{2i}$ $(i = 1, \ldots, N)$. At turn $2K-1$, Alice plays $x_{2K-1} = L - S_{2N}$. Since $S_{2N} \le N\sqrt{2}$ and $L > K\sqrt{2} = (N+1)\sqrt{2}$, $x_{2K-1} > \sqrt{2} > 0$.

Alice wins if Bazza cannot move at turn $2K$, i.e., $Q_{2K-1} > 2K$. $Q_{2K-1} = Q_{2N} + (L - S_{2N})^2$.

Bazza aims to minimize this quantity subject to his constraints: $y_i \ge 0$ and $\sum_{j=1}^{i} y_j^2 \le 2i$. These constraints imply $Q_{2N} \le 2N$, and consequently $S_{2N} \le N\sqrt{2}$.

Let $F(y) = Q_{2N}(y) + (L - S_{2N}(y))^2$. Consider the strategy $y^* = (\sqrt{2}, \ldots, \sqrt{2})$. This is feasible for Bazza as $\sum_{j=1}^{i}(\sqrt{2})^2 = 2i$. Let $S^* = N\sqrt{2}$ and $Q^* = 2N$.

Let y be any feasible strategy for Bazza. Let $\Delta S = S^* - S_{2N}(y) \ge 0$. We compare $F(y)$ with $F(y^*)$. We use the identity $\sum(y_i - \sqrt{2})^2 = Q_{2N}(y) - 2\sqrt{2}S_{2N}(y) + 2N$. $Q_{2N}(y) - Q^* = Q_{2N}(y) - 2N = \sum(y_i - \sqrt{2})^2 + 2\sqrt{2}S_{2N}(y) - 4N$. $2\sqrt{2}S_{2N}(y) = 2\sqrt{2}(S^* - \Delta S) = 4N - 2\sqrt{2}\Delta S$. $Q_{2N}(y) - Q^* = \sum(y_i - \sqrt{2})^2 - 2\sqrt{2}\Delta S$.

$F(y) - F(y^*) = Q_{2N}(y) - Q^* + (L - S_{2N}(y))^2 - (L - S^*)^2$. $(L - S_{2N}(y))^2 = (L - (S^* - \Delta S))^2 = (L - S^*)^2 + 2(L - S^*)\Delta S + (\Delta S)^2$.

$F(y) - F(y^*) = \left(\sum(y_i - \sqrt{2})^2 - 2\sqrt{2}\Delta S\right) + 2(L - S^*)\Delta S + (\Delta S)^2$. $F(y) - F(y^*) = \sum(y_i - \sqrt{2})^2 + 2(L - S^* - \sqrt{2})\Delta S + (\Delta S)^2$.

By the choice of K, $L > K\sqrt{2} = (N+1)\sqrt{2} = S^* + \sqrt{2}$. Let $\epsilon = L - S^* - \sqrt{2} > 0$. $F(y) - F(y^*) = \sum(y_i - \sqrt{2})^2 + 2\epsilon\Delta S + (\Delta S)^2$. Since all terms are non-negative, $F(y) \ge F(y^*)$. The minimum value of Q_{2K-1} is $F(y^*)$.

$Q_{2K-1} \ge F(y^*) = 2N + (L - N\sqrt{2})^2 = 2N + (\sqrt{2} + \epsilon)^2$. Since $\epsilon > 0$, $(\sqrt{2} + \epsilon)^2 > 2$. $Q_{2K-1} > 2N + 2 = 2K$. Bazza cannot move at turn $2K$. Alice wins.

Case 3: $\lambda = \frac{1}{\sqrt{2}}$. Neither player has a winning strategy.

We show that both players have a strategy to ensure the game continues forever (a draw).

1. Alice's drawing strategy (A-Zero): $x_{2k-1} = 0$ for all k. We verify the game continues forever. Alice's feasibility at turn $2k-1$: We need $S_{2k-2} \leq \lambda(2k-1)$. Bazza maximizes S_{2k-2} subject to $Q_{2k-2} \leq 2k-2$, achieving at most $(k-1)\sqrt{2}$. We check: $(k-1)\sqrt{2} \leq \frac{1}{\sqrt{2}}(2k-1) \iff 2k-2 \leq 2k-1$. True. Bazza's survival at turn $2k$: We need $Q_{2k-1} \leq 2k$. $Q_{2k-1} = Q_{2k-2} \leq 2k-2 < 2k$. Bazza survives. Alice's survival at turn $2k+1$: We need $S_{2k} \leq \lambda(2k+1)$. Bazza maximizes S_{2k} subject to $Q_{2k} \leq 2k$, achieving at most $k\sqrt{2}$. We check: $k\sqrt{2} \leq \frac{1}{\sqrt{2}}(2k+1) \iff 2k \leq 2k+1$. True. The game continues forever. Bazza cannot win.

2. Bazza's drawing strategy (B-MaxQ): $Q_{2k} = 2k$. We verify the game continues forever. Bazza's feasibility (survival). As shown in Case 1, if Bazza follows B-MaxQ, $S_{2k-2} \geq (k-1)\sqrt{2}$. Alice's budget $C_k = \lambda(2k-1) - S_{2k-2}$. $C_k \leq \frac{1}{\sqrt{2}}(2k-1) - (k-1)\sqrt{2} = \frac{2k-1-2(k-1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. Alice must choose $x_{2k-1} \leq 1/\sqrt{2}$. Then $Q_{2k-1} = 2k-2+x_{2k-1}^2 \leq 2k-2+1/2 < 2k$. B-MaxQ is feasible. Bazza survives.

Alice's survival. We must show $C_k > 0$ for all k. $C_1 = 1/\sqrt{2} > 0$. $C_{k+1} = C_k + 2\lambda - (x_{2k-1} + x_{2k}) = C_k + \sqrt{2} - (x_{2k-1} + x_{2k})$. Bazza ensures $x_{2k-1}^2 + x_{2k}^2 = 2$. Let $g(t) = t + \sqrt{2-t^2}$. $C_{k+1} = C_k + \sqrt{2} - g(x_{2k-1})$. Alice chooses $x_{2k-1} \in [0, C_k]$. To ensure Alice survives, we check the minimum possible budget for the next turn. Since $C_k \leq 1/\sqrt{2} < 1$ and $g(t)$ is increasing on $[0,1]$ (as $g'(t) = 1 - t/\sqrt{2-t^2} > 0$ for $t < 1$), $g(x_{2k-1})$ is maximized when $x_{2k-1} = C_k$. $C_{k+1} \geq C_k + \sqrt{2} - g(C_k) = \sqrt{2} - \sqrt{2 - C_k^2}$. If $C_k > 0$, then $\sqrt{2-C_k^2} < \sqrt{2}$, so $C_{k+1} > 0$. By induction, $C_k > 0$ for all k. Alice survives. The game continues forever. Alice cannot win.

Conclusion: Alice has a winning strategy if and only if $\lambda > \frac{1}{\sqrt{2}}$. Bazza has a winning strategy if and only if $0 < \lambda < \frac{1}{\sqrt{2}}$. If $\lambda = \frac{1}{\sqrt{2}}$, neither player has a winning strategy.